$|a_n - L| < \varepsilon \quad n > N(\varepsilon)$

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Test for Divergence: if a_n does not $\to 0$, it diverges

Geometric? $\sum_{n=1}^{\infty} a_1 r^{n-1}$, Sum is $\frac{a_1}{1-r}$ converges for $|r| < 1$

P-series? $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, otherwise diverges

Integral if $\int_1^\infty f(x)$ is convergent, $\sum_{n=1}^{\infty} a_n$ is convergent

if $\int_1^\infty f(x)$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent

estimate $R_n = s - s_n$, $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^\infty f(x)\,dx$

$R_n \leq$ accuracy $\quad s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq \int_n^\infty f(x)\,dx + s_n$

Comparison: $\sum a_n$ and $\sum b_n$ are both ≥ 0 for all n

if $a_n \leq b_n$ $\quad$ if a_n diverges, b_n diverges

if b_n converges, a_n converges

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then both converge or both diverge

Alternating series

alternating if $a_n a_{n+1} < 0$

let $b_n = |a_n|$ if $b_{n+1} \leq b_n$ for all n

then converges, also $\lim_{n\to\infty} b_n = 0$ if converges (use test for divergence on b_n)

estimate $|R_n| = |s - s_n| \leq b_{n+1}$

$b_{n+1} \leq$ accuracy

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Ratio Test

i) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, if $L < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

ii) , if $L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, series diverges

Root Test

i) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

ii) $L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then series is divergent

- no info at $L=1$ for Ratio or Root tests

Power Series

$\sum_{n=0}^{\infty} c_n (x-a)^n$

Three possibilities, i) series converges only when $x=a$

ii) series converges for all x

iii) series converges if $|x-a| < R$

diverges if $|x-a| > R$

$R = 1/L$, if $L=0$, R is all values of x test boundaries

Taylor and Maclaurin series

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ where $|x-a| < R$ $\quad c_n = \frac{f^{(n)}(a)}{n!}$

Maclaurin series where $a=0$

Binomial Series

$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, where $\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$

for $n \geq 1$, and $\binom{k}{0} = 1$

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Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\left.\begin{matrix} e^x \\ \sin x \\ \cos x \end{matrix}\right\} \text{ for } (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \Big\} \text{ for } [-1, 1]$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Big\} \text{ for } (-1, 1)$$

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Separable equation

$$\frac{dy}{dx} = h(y)g(x) \longrightarrow \int \frac{dy}{h(y)} = \int g(x)dx$$

note: $y(x) = G(x) + \underline{\underline{C}}$

Homogeneous Equation

$\frac{dx}{dx} = f(x,y)$ where $f(x,y)$ can be written as $g\left(\frac{y}{x}\right)$

make separable by letting $v = \frac{y}{x}$, then $y = vx$
$y' = v + xv'$

First Order Linear Equation

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \mu = e^{\int P(x)dx}$$

$$y = \frac{1}{\mu}\left(\int \mu Q(x)dx + C\right)$$

First Order Exact Equation

$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$ is exact if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

① then, integrate P or Q in terms of x or y (gives constant $g(y)$ or $g(x)$),
② then, differentiate $\int P dx$ or $\int Q dy$ in terms of y or x
③ this will give $g'(y)$ or $g'(x)$ which is = to Q or P
④ giving what $g'(y)$ or $g'(x)$ is
⑤ then integrate or, for equation ①

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Note: if <u>not</u> exact, use integrating factor

$$(I)\left(\frac{P_y - Q_x}{Q}\right) = \frac{dI}{dx}$$, solve for I

multiply equation by I. it is now exact (do a test too)

<u>Second Order Linear Equation</u>

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \qquad (1)$$

if $G(x)=0$, then homogeneous, if $G(x)\neq 0$, then non-homogeneous

<u>if homogeneous / constant coefficient</u>

$y = c_1 y_1(x) + c_2 y_2(x)$, let $y = e^{rx}$ in eqn (1)

then $ar^2 + br + c = 0$, solve for r

if discriminant, $D = b^2 - 4ac > 0$, then

$y_1(x) = e^{r_1 x}$

$y_2(x) = e^{r_2 x}$

if $D = 0$, then $r_1 = r_2$, $\therefore$

$y_1(x) = e^{rx}$

$y_2(x) = xe^{rx}$

if $D < 0$, then $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

$y_1(x) = e^{r_1 x}$

$y_2(x) = e^{r_2 x}$

note: $e^{i\theta} = \cos\theta + i\sin\theta$, $\therefore$ $y_1 = e^{\alpha x}(\cos\beta x + i\sin\beta x)$

$y_2 = e^{\alpha x}(\cos\beta x - i\sin\beta x)$

so $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

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Non-Homogeneous Linear Equations

if constant coefficients, $ay'' + by' + cy = G(x)$ $\boxed{y(x) = y_p + y_c}$

complementary eqn $ay'' + by' + cy = 0$

let $y = e^{rx}$, $ar^2 + br + c = 0$, find r

solve for y_c like is done for the homogeneous way

then make up y_p

$$\left.\begin{array}{l}
\begin{cases} \text{if } G(x) = C_n x^n + \dots + C_1 x + C_0 \\ \text{try } y(p) = A_n x^n + \dots + A_1 x + A_0 \end{cases} \\
\begin{cases} \text{if } G(x) = Ce^{kx} \\ \text{try } y(p) = Ae^{kx} \end{cases} \\
\begin{cases} \text{if } G(x) = C\cos kx + D\sin kx \\ \text{try } y(p) = A\cos kx + B\sin kx \end{cases}
\end{array}\right\}$$

if any term of y_p is in y_c, multiply y_p by x (or x^2, etc)

put y_p'', y_p', y_p into equation, solve for constants

Variation of parameters (2nd order, non-homogeneous, constant coefficient)

find $y_c = c_1 y_1 + c_2 y_2$ from $ay'' + by' + cy = 0$

then $u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = \frac{G(x)}{a}$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ G(x)/a \end{bmatrix}$$

$u_1' = \frac{[D_1]}{[D]}$

$u_2' = \frac{[D_2]}{[D]}$

solve for u_1, u_2 then $y_p = u_1 y_1 + u_2 y_2$

$y = y_c + y_p$

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<u>series solutions</u> 2nd order / non-constant coefficient

assume $y = \sum_{n=0}^{\infty} c_n x^n$, find y', y'', put indexes to the same value

put into eqn, put together

then the constant stuff $= 0$

write out constants for $n = 0, 1, 2, 3,$ etc

find a pattern (in terms of first few c_n s)

put back into eqn. $y = c_0 + c_1 x + c_2 x^2 + \ldots$

simplify - is it maclaurin? Yes - simplify